Clubs and the Market

Large Finite Economies

Ellickson, Bryan; Grodal, Birgit; Scotchmer, Suzanne; Zame, William

Publication date:
1999

Document version
Early version, also known as pre-print

Citation for published version (APA):
Ellickson, B., Grodal, B., Scotchmer, S., & Zame, W. (1999). *Clubs and the Market: Large Finite Economies*. Department of Economics, University of Copenhagen.

DISCUSSION PAPERS
Department of Economics
University of Copenhagen

99-05

Clubs and The Market:
Large Finite Economies

Bryan Ellickson, Birgit Grodal,
Suzanne Scotchmer, William R. Zame

Studiestræde 6, DK-1455 Copenhagen K., Denmark
Tel. +45 35 32 30 82 - Fax +45 35 32 30 00
http://www.econ.ku.dk

Clubs and the Market: Large Finite Economies[1]

Bryan Ellickson
University of California, Los Angeles

Birgit Grodal
University of Copenhagen

Suzanne Scotchmer
University of California, Berkeley

William R. Zame
University of California, Los Angeles

April 1997, revised January, 1999

Keywords: clubs, ε-cores, approximate equilibrium, decentralization.

JEL Classification: D2, D5, H4

[1]We thank Robert Anderson for tutelage, and Kenneth Arrow, Peter Hammond and Joe Ostroy, as well as many other seminar participants, for comments. We thank UC Berkeley, UCLA, the NSF, the Fulbright Foundation, and the Danish Social Science Research Council for financial support. We thank the UCLA and UC Berkeley Departments of Economics, and the Institute of Economics, University of Copenhagen for gracious hospitality during preparation of this paper.

Abstract We study large finite club economies in which agents can belong to several clubs, and care about the characteristics of the other club members. Club memberships must be integer consistent in aggregate. We show that states in the approximate core can approximately be decentralized by prices for private goods and for club memberships, that the approximate core is nonempty, and that approximate club equilibria exist. Our arguments use the convexification tools used for private goods economies, but we also develop a new tool to address the consistency requirement on memberships that are special to club economies. This tool allows us to overcome the integer consistency problems that are avoided in our (1999) paper by assuming a continuum of agents.

1 Introduction

It is well known that for large finite economies with indivisibilities, including club economies, the core might be empty, and equilibrium might not exist. In Ellickson, Grodal, Scotchmer and Zame (EGSZ 1999), we avoided these problems by assuming that there is a continuum of agents. In this paper we develop a tool that helps us to analyze large finite club economies. We show that states in the approximate core can approximately be decentralized by prices for private goods and club memberships, that the approximate core is nonempty, and that approximate club equilibria exist.

Similar approximation results have long been known for private goods economies, and rely on approximate versions of Lyapunov's convexity theorem. The club problem adds a new dimension, namely, the problem of ensuring that club memberships are consistent in the sense described below. To address the problem of consistency, we develop an essential tool, namely, Lemma 3.1.

As in EGSZ (1999), we describe a club type as a pair consisting of a description of the external characteristics of its members and a specified activity. We assume that there is a finite number of club types, each having a finite number of members, but many clubs of each type may be formed.[2] We also assume that the number of clubs an agent can join is bounded.

Intuitively, the boundedness assumptions should lead to competitiveness in a large economy. This is for two reasons. Individual agents have limited market power because they only have external effects on a bounded number of other agents, and clubs have limited market power because they could be duplicated by many other clubs of the same type. We validate these intuitions by showing that states in the approximate core (hence, the core) can approximately be decentralized by prices for private goods and club memberships.

We define a club membership as an opening in a club available to agents with specified characteristics, and treat these memberships as commodities in parallel to private goods. This parallel treatment leads naturally to trade and pricing of individual memberships, and easily accommodates that agents can have memberships in many clubs.[3] Despite this parallelism, club memberships introduce an important complexity, namely, that they must be consistent across the population. The allocation of club memberships must be such that all places in a club are filled as soon as just one member is assigned to it. Consistency must hold simultaneously for all types of clubs, and allow that every individual belongs to several clubs. In our finite economy, the consistency condition is particularly hard to satisfy, as compared to the continuum, because the number of clubs of each type must be an integer. As mentioned above, this might lead to nonexistence of equilibrium and emptiness of the core (Ellickson (1973), Bewley (1981), Scotchmer (1997)).

[2]Thus any club type is small relative to a large economy. This is the basic idea underlying club theory. The idea that optimal consumption groups are small was originally derived by Buchanan (1965) from congestion costs.

[3]In the previous literature, the admissions price to a single club is constructed as a willingness to pay. It is not obvious how to extend this technique to multiple memberships.

Also, in private goods economies the core and the set of equilibria might be empty when agents do not have convex preferences. This has led to notions of approximate core states and approximate equilibria (see e.g. Kannai (1970), Grodal and Hildenbrand (1974) and Anderson (1985)), theorems on existence of equilibrium states according to these notions (Starr (1969), Hildenbrand, Schmeidler, and Zamir (1973), Grodal (1974), Grodal, Trockel and Weber (1984)), and to decentralization theorems for elements in the approximate cores (see e.g. Grodal and Hildenbrand (1974), and Anderson (1985)). In this paper we define the fat ε-core for club economies in parallel to Anderson (1985) for private goods economies. This notion of approximate core is based on the requirement that it costs a per-capita amount ϵ of each commodity to form a blocking coalition. If $\epsilon = 0$, the fat ϵ-core is the core. For a given $\varepsilon \geq 0$, any state in the fat ε-core can approximately be decentralized in the sense that there exist prices for private commodities and for the club memberships such that all clubs have balanced budgets and such that the agents' consumptions are "demand-like" on average. The degree of approximation only depends on ε and some bounds on the economy, is independent of the number of agents, and converges to ε at the rate $\frac{1}{|A|}$, where A is the set of agents. Hence our result is the analogue for club economies to Anderson's (1978,1985) theorems for private goods economies.

Our theorem implies that the strong and weak ε-cores defined by Kannai (1970), as well as the core itself, can similarly be decentralized. However we prove the decentralization theorem for the fat ε-core because, as we show in a simple example, these other cores might easily be empty.

In order to prove that the fat ε-core is nonempty for a large economy (and also for independent interest), we define a notion of approximate equilibrium, show that an approximate equilibrium exists for all economies, and then show that all approximate equilbrium allocations are contained in the fat ε-core if the economy is sufficiently large. In the approximate equilibrium, all agents except at most a given number are optimizing. Hence, in large finite economies, an approximate equilibrium differs from a club equilibrium only in the sense that a fixed number of agents are not optimizing, but their proportion is negligible.

As mentioned above, the main difficulty in extending our results from the continuum case to large finite economies is in ensuring that club memberships are integer consistent. Our main tool to overcome this difficulty is Lemma 3.1: Starting from an arbitrary assignment of memberships, there is a bound on the number of agents that must be removed from the economy in order to guarantee that the memberships of the remaining agents are integer consistent. This bound depends on how far from consistent the original assignment was, but does not depend on the number of agents in the economy.

The description of the club economy is in Section 2. The main lemma is in Section 3. In Section 4 we define the fat-ϵ core and show that it can be approximately decentralized. In Section 5 we introduce our notion of approximate equilibrium, and use it to show nonemptiness of the fat-ϵ core. Proofs are collected in Section 6.

2 Club Economies

2.1 Private goods

Throughout, that there are $N \geq 1$ perfectly divisible, publicly traded private goods; thus the space of private goods is $\mathbb{R}^N_+$. For $x, x' \in \mathbb{R}^N_+$ we write $x \geq x'$ to mean $x_n \geq x'_n$ for each n, $x > x'$ to mean that $x \geq x'$ but $x \neq x'$, and $x >> x'$ to mean that $x_n > x'_n$ for each n. We write $|x| = \sum_{n=1}^N |x_n|$, and we let $\Delta = \{p \in \mathbb{R}^N_+ : |p| = 1\}$.

2.2 Clubs

Clubs are defined as in EGSZ (1999), where the model is discussed in more detail.

We will describe a *club type* by the number and characteristics of its members and the activity in which the club is engaged.

Let Ω be a finite set of *external characteristics* of potential members of a club. An element $\omega \in \Omega$ is a complete description of the external characteristics of an individual that are relevant for the other members of a club.

A *profile* (of a club) is a function $\pi : \Omega \to \mathbf{Z}_+ = \{0, 1, \ldots\}$ describing the external characteristics in a club. For $\omega \in \Omega$, $\pi(\omega)$ represents the number of members of the club having external characteristic ω. For a club with profile π, $|\pi| = \sum_{\omega \in \Omega} \pi(\omega)$ is the total number of members in the club.

There is a finite set Γ of *activities* available to a profile of agents.

A *club type* is a pair (π, γ) consisting of a profile and an activity $\gamma \in \Gamma$. We take as given a finite set of possible club types $\mathbf{Clubs} = \{(\pi, \gamma)\}$. In particular, there is a bound on the number of members of a club. Formation of the club (π, γ) requires a total input of private goods equal to $\mathbf{inp}(\pi, \gamma) \in \mathbb{R}^N_+$.

A *club membership* is an opening in a particular club type for an agent with a particular external characteristic; i.e., a triple $m = (\omega, \pi, \gamma)$ such that $(\pi, \gamma) \in \mathbf{Clubs}$ and $\pi(\omega) \geq 1$. Write $\mathcal{M}$ for the set of club memberships.

Each agent may belong to many clubs or to none. A *list* is a function $\ell : \mathcal{M} \to \{0, 1, \ldots\}$, where $\ell(\omega, \pi, \gamma)$ specifies the number of memberships of type (ω, π, γ). Write:

$$\mathbf{Lists} = \{\ell : \ell \text{ is a list }\}$$

for the set of lists. **Lists** is a set of functions from $\mathcal{M}$ to $\{0, 1, \ldots\}$, but we frequently view it as a subset of $\mathbb{R}^{\mathcal{M}}$, which is the set of functions from $\mathcal{M}$ to $\mathbb{R}$.

For an arbitrary list ℓ we define:

$$\tau(\ell) = \sum_{(\omega,\pi,\gamma)\in\mathcal{M}} \ell(\omega, \pi, \gamma) \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma) \quad \in \mathbb{R}^N_+$$

This is the total bundle of inputs that an agent consuming list ℓ would have to contribute if inputs to clubs were imputed equally to all members of all clubs.

2.3 Agents

A complete description of an agent a consists of his/her external characteristics, choice set, endowment of private goods, and utility function. An external characteristic is an element $\omega_a \in \Omega$. The choice set X_a specifies the feasible bundles of private goods and club memberships, so $X_a \subset \mathbb{R}_+^N \times \mathbf{Lists}$. For simplicity, we assume that the only restriction on private good consumption is that it is non-negative, so that $X_a = \mathbb{R}_+^N \times \mathbf{Lists}(a)$ for some subset $\mathbf{Lists}(a) \subset \mathbf{Lists}$. We assume that for all $\ell \in \mathbf{Lists}(a)$ we have $\ell(\omega, \pi, \gamma) = 0$ for every $(\omega, \pi, \gamma) \in \mathcal{M}$ for which $\omega \neq \omega_a$; that is, no individual may choose membership in any club type containing no members of his/her external characteristic. Moreover we assume that $0 \in \mathbf{Lists}(a)$. The endowment is denoted $e_a \in \mathbb{R}_+^N$. The utility function is defined over private goods consumptions and club memberships and is thus a mapping $u_a : X_a \to \mathbb{R}$. We assume throughout that utility functions $u_a(\cdot, \ell)$ are continuous and strictly monotone in private goods.

2.4 Club Economies

A *club economy* $\mathcal{E}$ is defined by a finite set of private commodities, a finite set of club types and their inputs, a finite set of agents A, and, finally, a mapping $a \mapsto (\omega_a, X_a, e_a, u_a)$ that assigns to each agent $a \in A$ his external characteristic, choice set, endowment and utility function.

We shall consider classes of economies where each class is identified according to the number of private commodities N, by the number of club memberships $|\mathcal{M}|$, by a bound M^* on the maximum number of members of any club type, by a bound W on individual endowments, that is, $e_a \leq W\mathbf{1}$

for all agents in the economy, and by a bound M on the number of club memberships an individual may choose. Let $\mathbf{Lists}_M = \{\ell \in \mathbf{Lists} : |\ell| \leq M\}$, then $\mathbf{Lists}(a) \subset \mathbf{Lists}_M$ for all $a \in A$.

We write $\mathbf{Econ}(N, |\mathcal{M}|, M^*, W, M)$ for the club economies with the stated bounds. Clearly these bounds also give a bound on the number of elements in $\mathbf{Lists}_M$ and on the number of external characteristics such that $\pi(\omega) > 1$ for some club type (π, γ). Without loss of generality we identify Ω with these external characteristics. Hence $|\Omega| \leq |\mathcal{M}|$.

We say that *endowments are desirable* in a club economy if for every agent $a \in A$ and for all $\ell \in \mathbf{Lists}(a)$, $u_a(e_a, 0) > u_a(0, \ell)$. This condition will be used in our existence argument.

2.5 Feasible States

A *state* of a club economy $\mathcal{E}$ is (x, μ) where $(x, \mu) : A \to \mathbb{R}^N \times \mathbb{R}^{\mathcal{M}}$ represents agents' consumptions.

Individual feasibility means $(x_a, \mu_a) \in X_a$ for all $a \in A$.

Social feasibilty entails market clearing for private goods and consistent matching of agents. We define consistency as a property of membership vectors, $\bar{\mu} \in \mathbb{R}^{\mathcal{M}}$. We say that an aggregate membership vector $\bar{\mu} \in \mathbb{R}^{\mathcal{M}}$ is *integer consistent* if for every club type $(\pi, \gamma) \in \mathbf{Clubs}$, there is a nonnegative integer $\alpha(\pi, \gamma)$ such that

$$\bar{\mu}(\omega, \pi, \gamma) = \alpha(\pi, \gamma)\, \pi(\omega), \qquad \text{each } \omega \in \Omega$$

The coefficient $\alpha(\pi, \gamma)$ is the number of clubs of type (π, γ) accounted for by the aggregate membership vector $\bar{\mu}$. Define

$$\mathbf{Cons}^* = \{\ \bar{\mu} \in \mathbb{R}^{\mathcal{M}} \ :\ \bar{\mu} \text{ is integer consistent }\}$$

We say that a list assignment $\mu : B \to \mathbf{Lists}$ is *integer consistent for B* if $\sum_{a \in B} \mu_a$ is integer consistent, *i.e.*, $\sum_{a \in B} \mu_a \in \mathbf{Cons}^*$.

In the following we will also use **Cons**, defined as the linear subspace spanned by $\mathbf{Cons}^*$.

Definition 2.1 *A state* (x, μ) *is feasible for* $B \subset A$ *if it satisfies the following requirements:*

(i) (individual feasibility) $(x_a, \mu_a) \in X_a$ for each $a \in B$

(ii) (material balance) $\sum_{a \in B} x_a + \sum_{a \in B} \tau(\mu_a) = \sum_{a \in B} e_a$

(iii) (integer consistency) $\sum_{a \in B} \mu_a \in \mathbf{Cons}^*$

We say the state (x, μ) is *feasible* if it is feasible for the set A itself.

Notice that by integer consistency the agents in B form $\alpha(\pi, \gamma)$ clubs of type (π, γ) for some non-negative integers $\alpha(\pi, \gamma)$. The definition of τ therefore implies that $\sum_{a \in B} \tau(\mu_a) = \sum_{(\pi,\gamma)} \alpha(\pi, \gamma)\mathbf{inp}(\pi, \gamma)$. Thus (ii) states that B's total endowments equals the sum of B's private consumption and the total input that B uses in the clubs they form.

A state of the economy will generally have several clubs of each club type. Since a state assigns the same inputs to all clubs of the same type, and since members care only about the external characteristics and not the identities of members, a list assignment μ does not assign membership in particular clubs, but only in particular club types.

We will use the following notation. For a given family $(y_a)_{a \in B}$ of vectors in a Euclidean space, we let $y_B \equiv \sum_{a \in B} y_a$, in particular, for a state (x, μ) and $B \subset A$, $x_B = \sum_{a \in B} x_a$ and $\mu_B = \sum_{a \in B} \mu_a$. The number of consumers in a coalition $B \subset A$ is denoted $|B|$.

3 Preliminaries

The main tools developed in this paper are the following two lemmas, which underlie our proofs. The lemmas are proved in Section 6.

The following lemma implies that if a list assignment ν is almost consistent for a coalition B in the sense that the distance from ν_B to **Cons** is bounded, and if B is large, then there is a subset $B' \subset B$ such that ν is integer consistent for B' and the size of B' is comparable to B.

Lemma 3.1 *There exist positive constants K_1, K_2 such that if B is a finite set and $\nu : B \to \mathbf{Lists}_M$ is a function, then there is a subset $B' \subset B$ such that*

$$\nu_{B'} \in \mathbf{Cons}^*$$

and

$$|B \setminus B'| \leq K_1 \mathrm{dist}\,(\nu_B, \mathbf{Cons}) + K_2$$

The next lemma Part (a) says that if a list assignment μ is integer consistent for B, then B can be partitioned into coalitions that are bounded independent of μ and B, such that μ is integer consistent for each of them. Part (b) states an immediate consequence.

Lemma 3.2 *There exists $K_3 > 0$ such that if a list assignment $\mu : A \to \mathbf{Lists}_M$ is integer consistent for a coalition $B \subset A$, then*

(a) B can be partitioned into $B = \cup_{i \in I} B^i$ such that $|B^i| \leq K_3$ and μ is integer consistent for each B^i.

(b) if $B_1 \subset B$, then there exists B_2 such that $B_1 \subset B_2 \subset B$, $|B_2| \leq K_3 |B_1|$ and μ is integer consistent for B_2.

The constants K_1, K_2 and K_3 depend on $|\mathcal{M}|$, M^* and M, as can be seen from the proofs.

4 The Approximate Core: Decentralization

Since the core might be empty in finite club economies, we consider approximate cores. The literature contains several notions of approximate core, which differ in the resource cost of forming a blocking coalition. In the strong and weak ϵ-cores as defined by Kannai (1970), the cost is, respectively $\epsilon\mathbf{1}$ and $\epsilon|B|\mathbf{1}$, where $|B|$ is the size of the blocking coalition. Since both of these cores might be empty in even large finite club economies, we use the fat ε-core, due to Anderson (1985), where the cost of forming the blocking coalition is $\epsilon|A|\mathbf{1}$, where $|A|$ is the size of the economy. Obviously the fat-ϵ core contains the strong and weak ϵ-core, and therefore the theorem below, which states that elements in the fat-ϵ core can be decentralized, also implies that the strong and weak ϵ-cores can be decentralized.

The fat ϵ-core is defined as follows.

Definition 4.1 *For $\varepsilon \geq 0$ we say that a coalition $B \subset A$ can ε-capitablock a state (x, μ) if there is a state (y, ν) such that $u_a(y_a, \nu_a) > u_a(x_a, \mu_a)$ for all $a \in B$, ν is integer consistent for B, and $y_B + \tau(\nu_B) \leq e_B - \varepsilon|A|\mathbf{1}$. The fat ε-core of a club economy $\mathcal{E}$, denoted $\mathbf{C}_\varepsilon(\mathcal{E})$, is the set of all feasible states which cannot be ε-capitablocked.*

Obviously, for each fixed economy, the fat ε-core shrinks as ε becomes smaller.

The following example illustrates that the fat ϵ-core can be nonempty when the strong or weak ϵ-core is empty.

Example 1: Assume that $\Omega = \{\bar{\omega}\}$, and that there is only one club type (π, ν) with 2 members, so $\pi(\bar{\omega}) = 2$. Hence there is one type of membership, $\mathcal{M} = \{(\bar{\omega}, \pi, \nu)\}$. Let $\mathbf{inp}(\pi, \nu) = 0$. In addition there is one private good, so each agent's consumption set is $X = X_a = \mathbb{R}_+ \times \{\ell | \ell : \mathcal{M} \to \{0, 1\}\}$. Agents have identical endowments $e_a = 1$ and identical utility functions $u : X \to \mathbb{R}$, defined as follows for $x \geq 0$.

$$\begin{aligned} u(x, 0) &= 1 - e^{-x} \\ u(x, 1) &= 4x \end{aligned}$$

Let A be the set of agents. A state $(x_a, \mu_a)_{a \in A}$ is feasible if (x_a, μ_a) is equal to $(x, 1)$ or $(x, 0)$ for each $a \in A$, depending on whether the individual consumes a club membership or not, $\sum_{a\in A} x_a = \sum_{a\in A} e_a = |A|$, and $\sum_{a\in A} \mu_a = 2\alpha$ for some $\alpha \in Z_+$.

We now show that for all $\epsilon < \frac{1}{4}$, the weak ϵ-core is empty, irrespective of how large the economy is, provided $|A|$ is odd.

Let $(x_a, \mu_a)_{a\in A}$ be a feasible state. We show that the state can be weakly ϵ-blocked. Since there are an odd number of agents, at least one agent, say a_1, is not in a club ($\mu_{a_1} = 0$). Since he cannot weak ϵ-block, $x_{a_1} \geq 1-\epsilon$. Hence there exists an agent, say a_2, such that $x_{a_2} \leq 1+\epsilon$. Consider the coalition (a_1, a_2) and assume they form a club. If a_1 receives private good at least $\frac{1}{4}$, he receives at least as much utility as in the feasible state. If a_2 receives at least $1+\epsilon$ private goods, he receives at least as much utility as in the feasible state, whether or not he was in a club in the feasible state. They can weak ϵ-block, since $\frac{1}{4} + (1+\epsilon) < 2 - 2\epsilon$. Hence the weak ϵ-core is empty.

In contrast, the fat ϵ-core is nonempty for any $\epsilon > 0$, provided the set of agents is large enough. The easiest example of an allocation in the fat ϵ-core is to let each agent consume his endowment, and to form as many clubs of size 2 as possible, with one leftover agent. Write f^* for an allocation of this type. For any $\epsilon > 0$, f^* is in the fat ϵ-core if $|A| \geq \frac{1}{\epsilon}(\frac{3e+1}{4e})$.

However, for later use we define a different feasible state f^ϵ which is in the fat-ϵ core for $|A| \geq \frac{1}{\epsilon} + 1$. Let $(x_a, \mu_a) = (1-\epsilon, 1)$ for $a \neq a_1$, and $(x_{a_1}, \mu_{a_1}) = (1+\epsilon(|A|-1), 0)$. It is sufficient to consider a blocking coalition (a_1, a_2) for any a_2. Adding more agents to the blocking coalition would not improve the ability to block, since the added agents must consume at least as much private good in the blocking coalition as they consume in the feasible state f^ϵ, namely $1-\epsilon$. In order for the coalition (a_1, a_2) to fat ϵ-block f^ϵ, agent a_1 must consume at least $\frac{1}{4}(1 - e^{[-1-\epsilon(|A|-1)]})$ and a_2 must consume at least $1-\epsilon$, and they must pay $\epsilon|A|$ to form the coalition. But this is impossible since, using $|A| \geq \frac{1}{\epsilon} + 1$, we have $\frac{1}{4}(1 - e^{[-1-\epsilon(|A|-1)]})$ $+(1-\epsilon) + \epsilon|A| > 2$. ♣

We now extend Anderson's (1978,1985) approximate decentralzation theorems to the club context. We show that for large finite club economies, fat

ε-core states can approximately be supported by prices.

Club memberships must be priced as well as private goods, hence $(p, q) \in \mathbb{R}^N \times \mathbb{R}^{\mathcal{M}}$. Because we assume that preferences are monotone in private goods, private goods prices will be non-negative. However, prices for club memberships may be positive, negative or zero; prices for club memberships include transfers between agents in a given club — some agents may subsidize others.

In both the approximate decentralization and aproximate equilibrium we will require that prices are such that every club type has a balanced budget, that is, for each $(\pi, \gamma) \in \mathbf{Clubs}$, $\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \mathbf{inp}(\pi, \gamma)$. Clearly such membership prices can be decomposed into two parts, one part corresponding to the value of the member's share of the required inputs, $\frac{1}{|\pi|}\mathbf{inp}(\pi, \gamma)$, and additional 'pure transfers', say q^*, so-called because they transfer purchasing power within a club. The relationship between the membership prices q and pure transfers q^* is

$$q^*(\omega, \pi, \gamma) = q(\omega, \pi, \gamma) - p \cdot \frac{1}{|\pi|}\mathbf{inp}(\pi, \gamma)$$

Say that $q^* \in \mathbb{R}^{\mathcal{M}}$ are *pure transfer prices* if $q^* \in \mathbf{Trans}$, defined as:

$$\mathbf{Trans} = \{q^* \in \mathbb{R}^{\mathcal{M}} : q^* \cdot \mu = 0 \text{ for each } \mu \in \mathbf{Cons}^*\}$$

Clearly, $q^* \in \mathbf{Trans}$ if and only if $q^* \cdot \mu = 0$ for all $\mu \in \mathbf{Cons}$.

In the proofs we use the pure transfer prices instead of the equivalent membership prices, with budget sets appropriately redefined.

We now come to the decentralization theorem. Following Anderson, we define two measures of how well a given price system $(p, q) \in \Delta \times \mathbb{R}^{\mathcal{M}}$ approximately decentralizes a feasible state $f = (x, \mu)$ for an economy $\mathcal{E}$.

First, for $a \in A$ we define

$\rho_a^1(f, p, q) = [(p, q) \cdot (x_a, \mu_a) - p \cdot e_a]^+$

$\rho_a^2(f, p, q) = \sup\,[p \cdot e_a - (p, q) \cdot (x', \mu')]^+ : u_a(x', \mu') > u_a(x_a, \mu_a)\,]$

where we use the notation $r^+ = \max\{r, 0\}$. The number $\rho_a^1(f, p, q)$ measures how far (x_a, μ_a) lies outside agent a's budget set. The number $\rho_a^2(f, p, q)$ measures how much less than the value of his initial endowment the agent a may spend to get something that is preferred to (x_a, μ_a). Note that $\rho_a^1(f, p, q) = 0$ if and only if (x_a, μ_a) lies in agent a's budget set and that $\rho_a^2(f, p, q) = 0$ if and only if (x_a, μ_a) is a quasi-optimizing bundle for agent a.

The two measures of Anderson are then

$$\begin{aligned} \rho^1(f, p, q) &= \frac{1}{|A|} \sum_{a \in A} \rho_a^1(f, p, q) \\ \rho^2(f, p, q) &= \frac{1}{|A|} \sum_{a \in A} \rho_a^2(f, p, q) \end{aligned}$$

Definition 4.2 *A feasible state $f = (x, \mu)$ can be $\phi-$decentralized by prices if there exist $(p, q) \in \Delta \times \mathbb{R}^{\mathcal{M}}$ such that*

(i) $\rho^1(f, p, q) \leq \phi$,

(ii) $\rho^2(f, p, q) \leq \phi$ and

(iii) for each $(\pi, \gamma) \in$ **Clubs**, $\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \mathbf{inp}(\pi, \gamma)$.

Example 1 (continued): We now show that the feasible state f^ϵ can be approximately decentralized. Let the prices be $(p, q) = (1, 0)$. Then $\rho_a^1(f^\epsilon, p, q) = 0$ for all $a \neq a_1$, $\rho_{a_1}^1(f^\epsilon, p, q) = \epsilon(|A| - 1)$, $\rho_a^2(f^\epsilon, p, q) = \epsilon$ for all $a \neq a_1$, and $\rho_{a_1}^2(f^\epsilon, p, q) < 1$. Since $\rho^1(f^\epsilon, p, q) < \epsilon$, $\rho^2(f^\epsilon, p, q) < \epsilon + \frac{1}{|A|}$ and all clubs have balanced budgets, it follows that f^ϵ can be $(\frac{1}{|A|} + \epsilon)$-decentralized ♣

Theorem 4.3 *Let $\varepsilon \geq 0$. There is a constant K such that for any economy $\mathcal{E} \in$ **Econ**$(N, |\mathcal{M}|, M^*, W, M)$, and $(x^*, \mu^*) \in \mathbf{C}_\varepsilon(\mathcal{E})$, the state (x^*, μ^*) can be $(\frac{K}{|A|} + \varepsilon)$-decentralized by prices.*

The constant is $K = W(K_1(N+|\mathcal{M}|)M + K_2 + N + |\mathcal{M}|)$, where K_1 and K_2 are the constants from Lemma 3.1.

Remark 4.4 *The above theorem also applies to the states in the core, the strong ϵ-core and weak ϵ-core. However for states in the core and the strong ϵ-core, the degree of approximation can be improved.*

Corollary 4.5 *Let $\varepsilon \geq 0$. There is a constant K such that for any economy $\mathcal{E} \in \mathbf{Econ}(N, |\mathcal{M}|, M^*, W, M)$:*

(i) if (x^*, μ^*) is in the core of $\mathcal{E}$, then the state (x^*, μ^*) can be $\frac{K}{|A|}$-decentralized.

(ii) if (x^*, μ^*) is in the strong ϵ-core of $\mathcal{E}$, then the state (x^*, μ^*) can be $\frac{K+\epsilon}{|A|}$-decentralized.

The corollary follows trivially by letting $\varepsilon = 0$ to describe core states, and since the strong ϵ-core is contained in the fat $\frac{\epsilon}{|A|}$-core.

As in Anderson (1978), Theorem 3 is proved by separating the convex hull of the modified aggregate net preferred set from a translate of an appropriate cone. There are two subtleties:

(i) In Anderson (1978), the feasible cone from which the aggregate preferred set is separated is a translate of the negative orthant. The cone we use is the product of a translate of the negative orthant and the subspace **Cons** representing consistent membership choices. However in order to ensure that the price vector for the private goods is different from zero, we separate from a slightly enlarged cone.

(ii) To show that the convex hull of the modified aggregate net preferred set is disjoint from our translated cone, we will need to show that, if it were not, we would be able to construct a blocking coalition. To construct a blocking coalition in which all agents get a preferred bundle, we must throw out a few agents. When we do this, however, we may find that the membership choices of the remaining agents have become inconsistent. We use Lemma 3.1 to restore integer consistency of membership choices by throwing out still more agents.

5 The Approximate Core: Existence

In order to show that the above decentralization theorem is not vacuous, we show existence of states in the fat ϵ-core.

Theorem 5.1 *There is a positive integer $\bar{K}$ such that for any $\varepsilon > 0$ and any economy $\mathcal{E} \in \mathbf{Econ}(N, |\mathcal{M}|, M^*, W, M)$ with $|A| > \frac{\bar{K}}{\varepsilon}$ we have $\mathbf{C}_\varepsilon(\mathcal{E}) \neq \emptyset$.*

In order to prove the theorem we introduce a notion of approximate equilibrium, then show in Theorem 5.3 that such approximate equilibria exist, and then show in Theorem 5.4 that they are in the fat ϵ-core for a sufficiently large economy. Theorem 5.1 follows using the constant $\bar{K} = \hat{K}S$, where S and $\hat{K}$ are the constants in the Theorems 5.3 and 5.4.

For each non-negative integer S, we define an S-quasi-equilibrium consisting of a feasible state (x, μ) and prices $(p, q) \in \Delta \times \mathbb{R}^{\mathcal{M}}$. In addition to being feasible, the state satisfies: (i) all agents are in their budget sets, (ii) all agents except for at most S are consuming quasi-optimal bundles, and (iii) the prices are such that every club type has a balanced budget. Clearly, when the economy is large, i.e., $|A|$ is large relatively to S, an S-quasi-equilibrium is an approximate equilibrium in the sense that only relatively few agents are not quasi-optimizing.

Definition 5.2 *Let S be a positive integer. An S-quasi-equilibrium consists of a feasible state (x, μ) and prices $(p, q) \in \Delta \times \mathbb{R}^{\mathcal{M}}$, satisfying the following conditions:*

(i) (budget feasibility for individuals)

For every $a \in A$: $\quad (p,q)\cdot(x_a,\mu_a) \le p\cdot e_a$

(ii) (all except S agents optimize)

There is a set of agents $D \subset A$ such that

(a) $|D| \le S$;

(b) for $a \in A \setminus D, u_a(x'_a,\mu'_a) > u_a(x_a,\mu_a) \Rightarrow (p,q)\cdot(x'_a,\mu'_a) \ge p\cdot e_a$;

(iii) (budget balance for club types)

For each $(\pi,\gamma) \in \mathbf{Clubs}$, $\sum_{\omega\in\Omega}\pi(\omega)q(\omega,\pi,\gamma) = p\cdot\mathbf{inp}(\pi,\gamma)$

In Example 1, the feasible state f^*, together with prices $(p,q) = (1,0)$, is a 1-equilibrium.

Clearly an S-(quasi-)equilibrium is an S'-(quasi-)equilibrium for $S' \ge S$.

An S-equilibrium is defined in the obvious way; for the relationship with S-quasi-equilibrium, see EGSZ (1999).

Theorem 5.3 *There is a positive integer S such that an S-quasi-equilibrium exists for any $\mathcal{E} \in \mathbf{Econ}(N,|\mathcal{M}|,M^*,W,M)$ in which endowments are desirable.*

The integer S depends on the parameters of the class of economies according to

$$S = N + |\mathcal{M}| + K_1 M(N + 2|\mathcal{M}|) + K_2$$

where K_1 and K_2 are the constants defined in Lemma 3.1.

The proof of Theorem 5.3 is similar to the proof of existence of club equilibrium for the continuum in EGSZ (1999). However, since we now have a finite economy, the aggregate excess demand correspondence does not necessarily have convex values. Hence we must face directly the nonconvexity and integer consistency problems. These problems are overcome by using the Shapley-Folkmann theorem and especially Lemma 3.1. Having found a point in the convex hull of agents' excess demand functions, we use the Shapley-Folkman theorem to select a large subset of agents whose consumptions of private goods are in their demand, and then use Lemma 3.1 to find a smaller (but still large) subset whose club memberships are integer consistent. The number of agents excluded from the original economy is bounded independent of $|A|$.

We now turn to the connection between the set of S-equilibria and the fat ϵ-core.

Theorem 5.4 *There exists a constant $\hat{K}$ such that for any $\varepsilon > 0$ and any non-negative integer $S > 0$ and any economy $\mathcal{E} \in \mathbf{Econ}(N, |\mathcal{M}|, M^*, W, M)$ with $|A| > \frac{\hat{K}S}{\varepsilon}$ all S-quasi-equilibrium states for $\mathcal{E}$ belong to $\mathbf{C}_\varepsilon(\mathcal{E})$.*

The proof of Theorem 5.4 follows familiar lines, except that we must again use our basic lemmas, in particular Lemma 3.2, to achieve a contradiction from the assumption that a coalition can fat ϵ-block.

6 Proofs

The proofs of Lemma 3.1 and Lemma 3.2 rely on the following preliminary lemma.

We say that a non-empty subset $\mathcal{H} \subset \mathbf{Z}^n_+$ is *closed under addition and relative subtraction* if:

(i) $x, x' \in \mathcal{H} \Rightarrow x + x' \in \mathcal{H}$

(ii) $x, x' \in \mathcal{H}, x - x' \geq 0 \Rightarrow x - x' \in \mathcal{H}$

We say that $\mathcal{G} \subset \mathcal{H}$ *generates* $\mathcal{H}$ if $\sum_{y \in \mathcal{G}} n(y)y \in \mathcal{H}$ for every set of non-negative integers $\{n(y); y \in \mathcal{G}\}$, and if for every $x \in \mathcal{H}$ there exist non-negative integers $\{n_x(y); y \in \mathcal{G}\}$ such that $x = \sum_{y \in \mathcal{G}} n_x(y)y$.

Lemma 6.1 *Every nonempty subset $\mathcal{H} \subset \mathbf{Z}_+^n$ which is closed under addition and relative subtraction is generated by a finite set.*

Proof of Lemma 6.1 For each integer $k > 0$, write $\mathcal{H}_k = \{x \in \mathcal{H} : |x| \leq k\}$. We first establish the following claim:

> **Claim** There is an integer k such that every element of $\mathcal{H}$ dominates some nonzero element of $\mathcal{H}_k$. That is, for each $x \in \mathcal{H}$ there is a $y \in \mathcal{H}_k, y \not= 0$ such that $x \geq y$.

To prove the **Claim**, suppose the **Claim** is false. Then for each integer k there is an $x^k \in \mathcal{H}$ which does not dominate *any* element of $\mathcal{H}_k$. In particular, $x^k \notin \mathcal{H}_k$, so $|x^k| > k$. For each coordinate $1 \leq i \leq n$, the sequence (x_i^k) is either bounded or not. If it is bounded we may use the fact that elements of $\mathcal{H}$ have non-negative integer coordinates to extract a subsequence that is constant; if it is unbounded we may extract a subsequence that is strictly increasing to infinity. Applying the same reasoning to each coordinate in turn, we may extract a subsequence (x^{k_j}) that is non-decreasing; i.e., $x^{k_j} \leq x^{k_{j+1}}$ for each j. Set $k^* = |x^{k_1}|$. Because $k_j \to \infty$, there is an index j^* such that $k_{j^*} > k^*$. Since $x^{k_j} \geq x^{k_1}$ for every j, it follows that $x^{k_{j^*}} \geq x^{k_1}$. On the other hand, x^{k_1} is an element of $\mathcal{H}_{k^*}$ which is a subset of $\mathcal{H}_{k_{j^*}}$, so $x^{k_{j^*}}$ dominates an element of $\mathcal{H}_{k_{j^*}}$. This is a contradiction, so we obtain the **Claim**.

Now let $\mathcal{G} = \mathcal{H}_k$; clearly G is finite. We assert that $\mathcal{G}$ generates $\mathcal{H}$. Obviously, a combination of elements of $\mathcal{G}$ is in $\mathcal{H}$, by property (i). We must show that every element of $\mathcal{H}$ can be written as a non-negative integer combination of elements of $\mathcal{G}$. Note that

$$\mathcal{H} = \bigcup_{r=1}^{\infty} \mathcal{H}_r$$

Hence it suffices to show that for each r, every element of $\mathcal{H}_r$ can be written as a non-negative integer combination of elements of $\mathcal{G}$. To see this, suppose not. Then there is a smallest index r and an element of $\mathcal{H}_r$ which can can not be written as a non-negative integer combination of elements of $\mathcal{G}$. Certainly $r > k$ because $\mathcal{G} = \mathcal{H}_k$. Let $x \in \mathcal{H}_r$. By the **Claim**, there is a nonzero element $y \in \mathcal{G}$ with $x \geq y$. The hyotheses (ii) on $\mathcal{H}$ guarantees that $x - y \in \mathcal{H}$. Because $|x - y| < r$, minimality of r entails that $x - y$ can be written as a non-negative integer combination of elements of $\mathcal{G}$. Since $x = (x-y)+y$ and $y \in \mathcal{G}$, it follows that we can also write x as a non-negative integer combination of elements of $\mathcal{G}$, which contradicts the definition of r. ■

We now prove Lemma 3.1, which states that if a list assignment ν is almost consistent for B, then there is a large subset $B' \subset B$ for which ν is integer consistent. In the proof we work in two spaces: "membership space" and "list space" as shown in Figures 1 and 2. The list space has higher dimension than the membership space. A vector in the list space designates the number of lists of each type, whereas a vector in the membership space designates the number of memberships of each type. Consistency is a condition on vectors in the membership space (representing aggregate memberships). It is convenient to work in the list space because removing an agent from the set B corresponds to removing a list of memberships, not an individual membership.

Proof of Lemma 3.1 First, $\mathbf{Cons}^* \subset \mathbf{Z}_+^{\mathcal{M}}$ is closed under addition and relative subtraction; let $\mathcal{G}_1$ be a finite set of generators. Recall the notation

$$\mathbf{Lists}_M = \{\ell \in \mathbf{Lists} : |\ell| \leq M\}$$

We view $\mathbf{Lists}_M$ as the unit vectors in $\mathbb{R}^{\mathbf{Lists}_M}$, and define a linear map $T : \mathbb{R}^{\mathbf{Lists}_M} \to \mathbb{R}^{\mathcal{M}}$ by $T(\delta_\ell) = \ell$ for each unit vector δ_ℓ. Set

$$\mathcal{J} = \{x \in \mathbf{Z}_+^{\mathbf{Lists}_M} : T(x) \in \mathbf{Cons}^*\}$$

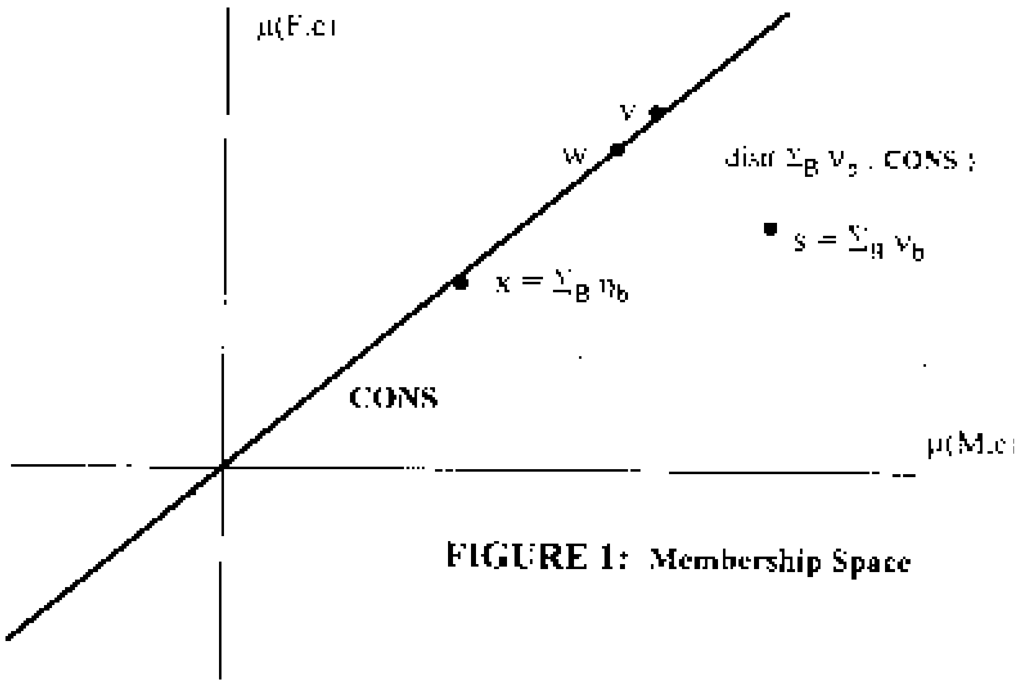
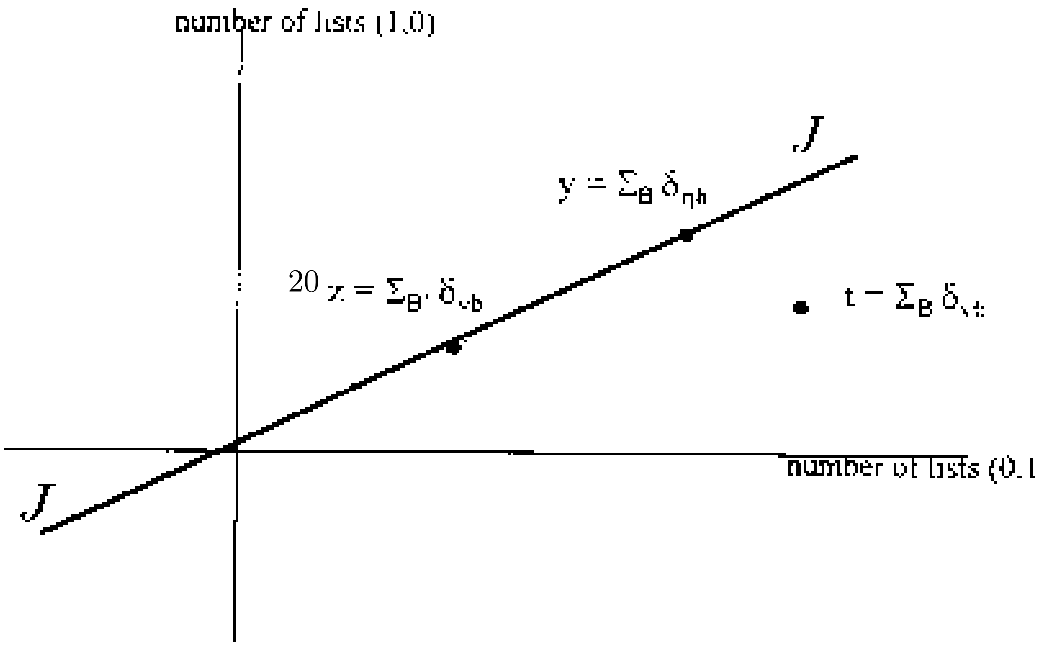


FIGURE 1: Membership Space



The set $\mathcal{J}$ describes consistency in $\mathbb{R}^{\mathbf{Lists}_M}$. It is easily checked that $\mathcal{J}$ is closed under addition and relative subtraction; let $\mathcal{G}_2$ be a finite set of generators.

Define constants K_1, K_2 by:

$$\begin{aligned} K_1 &= 2\left(\max_{g\in\mathcal{G}_1}|g|+1\right)\left(\max_{g'\in\mathcal{G}_2}|g'|+1\right) \\ K_2 &= K_1|\mathcal{M}|M^* \end{aligned}$$

Note that K_1, K_2 depend only on $|\Omega| \leq |\mathcal{M}|$, on the number of club types, and on the bound M on the number of memberships that may be chosen by an individual.

The proof now proceeds through several intermediate constructions and estimations. Write

$$s = \nu_B \in \mathbb{R}^{\mathcal{M}}$$

and

$$t = \sum_{b\in B} \delta_{\nu_b} \in \mathbb{R}^{\mathbf{Lists}_M}$$

The vector s represents the *memberships* assigned by ν to B, and t represents the *lists* assigned by ν to B. We will find $z \in \mathcal{J}$ such that $z \leq t$ and estimate $|t-z|$. The vector z also represents a set of lists, and $t-z$ represents the lists that are removed from t by removing members of B. We shall define B' such that $|B \setminus B'| = |t-z|$ and

$$\sum_{b\in B'} \delta_{\nu_b} = z \in \mathcal{J}$$

The definition of $\mathcal{J}$ and the definition and linearity of the mapping T entail that

$$\nu_{B'} = \sum_{b\in B'} T(\delta_{\nu_b}) = T\left(\sum_{b\in B'} \delta_{\nu_b}\right) \in \mathbf{Cons}^*$$

In order to construct z and estimate $|z-t|$, we first estimate $\mathrm{dist}\,(s, \mathbf{Cons}^*)$. We then construct an $x \in \mathbf{Cons}^*$ for which we can estimate $|x-s|$; this estimation is made easier by arranging that $x \leq s$. From this x we construct a $y \in \mathcal{J}$, and estimate $|y - t|$. Using y we construct the desired $z \in \mathcal{J}$ with $z \leq t$, and estimate $|z - t|$. See Figures 1 and 2.

Step 1 We estimate $\operatorname{dist}(s, \mathbf{Cons}^*)$. To this end, choose an element $v \in$ **Cons** such that

$$|s - v| = \operatorname{dist}(s, \mathbf{Cons})$$

Note that $v \geq 0$, for otherwise the positive part v^+ belongs to **Cons** (by the definition) and is closer to s (which is positive) than is v. By definition, for each club type (π, γ) there is a real number $\alpha(\pi, \gamma)$ such that for every $\omega \in \Omega$,

$$v(\omega, \pi, \gamma) = \alpha(\pi, \gamma)\pi(\omega)$$

Since $v \geq 0$ and $\pi \geq 0$, we have $\alpha(\pi, \gamma) \geq 0$ for each (π, γ). For each (π, γ) let $\bar{\alpha}(\pi, \gamma)$ be the greatest integer less than or equal to $\alpha(\pi, \gamma)$ and let $w \in \mathbf{Lists}_M$ be defined by

$$w(\omega, \pi, \gamma) = \bar{\alpha}(\pi, \gamma)\pi(\omega)$$

for each club (π, γ). This construction guarantees that $w \in \mathbf{Cons}^*$ and that

$$0 \leq v(\omega, \pi, \gamma) - w(\omega, \pi, \gamma) \leq \pi(\omega) \leq M^*$$

for each membership (ω, π, γ), so

$$|w - v| \leq |\mathcal{M}| M^*$$

Hence

$$\operatorname{dist}(s, \mathbf{Cons}^*) \leq |s - w| \leq |\mathcal{M}| M^* + \operatorname{dist}(s, \mathbf{Cons}) \tag{1}$$

Step 2 We construct an element $x \in \mathbf{Cons}^*$ that is dominated by s. If $w \leq s$, take $x = w$. If $w \not\leq s$, there is a membership $m \in \mathcal{M}$ such that $w(m) > s(m)$. Use Lemma 6.1 to write

$$w = \sum_{y \in \mathcal{G}_1} n_w(y) y$$

Pick $y^* \in \mathcal{G}_1$ such that $n_w(y^*) > 0$ and $y^*(m) > 0$; set

$$x^1 = [(n_w(y^*) - 1])y^* + \sum_{y \neq y^*} n_w(y)y \ \in \ \mathbf{Cons}^*$$

so that $x^1 \leq w$ and $x^1(m) < w(m)$. Continuing in this way we construct a decreasing sequence $x^1 \geq x^2 \ldots$ of elements of $\mathbf{Cons}^*$. After at most $|s - w|$ iterations, we obtain a vector $x \in \mathbf{Cons}^*$ with $x \leq s$. Since we subtract an element of $\mathcal{G}_1$ at each iteration, we conclude that

$$|w - x| \leq \left[\max_{g \in \mathcal{G}_1} |g|\right] |s - w| \tag{2}$$

Step 3 By definition, $s = \nu_B$ and $\nu_b \in \mathbf{Lists}_M$ for each b. We construct a function $\eta : B \to \mathbf{Lists}_M$ such that $\eta_b \leq \nu_b$ for each $b \in B$ and

$$\sum_{b \in B} \eta_b = x$$

To accomplish this, write

$$B = \{b_1, \ldots, b_n\}$$

Proceed inductively:

$$\begin{aligned}
\eta_{b_1} &= \min\{\nu_{b_1}, x\} \\
\eta_{b_2} &= \min\{\nu_{b_2}, x - \eta_{b_1}\} \\
&\vdots \\
\eta_{b_n} &= x - \sum_{1 \leq i \leq n-1} \eta_{b_i}
\end{aligned}$$

Step 4 Set

$$y = \sum_{b \in B} \delta_{\eta_b}$$

The definition of T implies that $T(y) = x$ so $y \in \mathcal{J}$. Write

$$B'' = \{b \in B : \eta_b = \nu_b\}$$

Because $0 \leq \eta_b \leq \nu_b$ for each $b \in B$ and $\nu_b - \eta_b \in \mathbf{Z}_+^{\mathcal{M}}$, it follows that

$$\begin{array}{rcrll} |\nu_b - \eta_b| & = & 0 & \text{if} & b \in B'' \\ |\nu_b - \eta_b| & \geq & 1 & \text{if} & b \in B \setminus B'' \end{array}$$

Note that $|\delta_{\nu_b} - \delta_{\eta_b}| = 2$ whenever $\nu_b \not= \eta_b$. Hence, since $t = \sum_{b \in B} \delta_{\nu_b}$,

$$|y - t| = 2|B \setminus B''|. \tag{3}$$

Moreover

$$|s - x| = \sum_{b \in B} |\nu_b - \eta_b| = \sum_{b \in B \setminus B''} |\nu_b - \eta_b| \geq |B \setminus B''| \tag{4}$$

Step 5 Proceeding exactly as in Step 2 we construct an element $z \in \mathcal{J}$ such that $z \leq t$ and

$$|z - y| \leq \left(\max_{g' \in \mathcal{G}_2} |g'| \right) |t - y| \tag{5}$$

Step 6 For each $\ell \in \mathbf{Lists}_M$, write

$$B_\ell = \{b \in B : \nu_b = \ell\}$$

By construction, $z \leq t$ so $z(\ell) \leq t(\ell) = |B_\ell|$ for each ℓ. Hence we may choose subsets $B'_\ell \subset B_\ell$ such that $|B'_\ell| = z(\ell)$. Setting

$$B' = \bigcup_\ell B'_\ell$$

therefore yields a subset $B' \subset B$ such that

$$\sum_{b \in B'} \delta_{\nu_b} = z \in \mathcal{J}$$

As noted at the beginning of the proof, linearity of T implies

$$\nu_{B'} = \sum_{b \in B'} T(\delta_{\nu_b}) = T(z) \in \mathbf{Cons}^*$$

Our construction implies that

$$|B \setminus B'| = \sum_{\ell \in \mathbf{Lists}_M} |B_\ell \setminus B'_\ell| = \sum_{\ell \in \mathbf{Lists}_M} |t(\ell) - z(\ell)| = |t - z| \qquad (6)$$

Combining (1) – (6), expanding, and substituting the definitions of K_1, K_2 yields the required estimate for $|B \setminus B'|$. ■

Proof of lemma 3.2(a): Let $T, \mathcal{J}$, and $\mathcal{G}_2$ be as defined in the proof of Lemma 3.1, and let

$$K_3 \equiv \max\ \{|g|\ : g \in \mathcal{G}_2\}. \qquad (7)$$

Now, let $\mu : B \to \mathbb{R}^{\mathcal{M}}$ be an integer consistent assignment for B. Hence $\sum_{a \in B} \delta_{\mu_a} \in \mathcal{J} \subset \mathbb{R}^{\mathbf{Lists}_{\mathcal{M}}}$. As $\mathcal{G}_2$ generates $\mathcal{J}$ there exists non-negative integers $\{n(y) : y \in \mathcal{G}_2\}$ such that $\sum_{a \in B} \delta_{\mu_a} = \sum_{y \in \mathcal{G}_2} n(y)y$. Now we disaggregate B in groups $B^j(y)$, where $y \in \mathcal{G}_2$ and $j = 1, \cdots, n(y)$, such that for each $B^j(y)$ we have $\sum_{a \in B^j(y)} \delta_{\mu_a} = y$. Formally, for each $\ell \in \mathbf{Lists}_M$ let $B_\ell = \{a \in B\, |\mu_a = \ell\}$. Hence we obtain $|B_\ell| = (\sum_{a \in B} \delta_{\mu_a})_\ell = \sum_{y \in \mathcal{G}_2} n(y)y_\ell$. Thus there exists a partion $\{B^j_\ell(y) : j = 1, ...n(y)$ and $y \in \mathcal{G}_2\}$ of B_ℓ such that $\left|B^j_\ell(y)\right| = y_\ell$ for all $j = 1, ..., n(y)$. Now define $B^j(y) = \bigcup_{\ell \in Lists_M} B^j_\ell(y)$. Then we obtain $\sum_{a \in B^j(y)} \delta_{\mu_a} = y$ for all $y \in \mathcal{G}_2$ and $j = 1, ..., n(y)$ and $\bigcup_{y \in \mathcal{G}_2} \bigcup_{j=1}^{n(y)} B^j(y) = B$.

Since $y \in \mathcal{G}_2 \subset \mathcal{J}$, we obtain that μ is integer consistent for all groups $B^j(y)$ and $|B^j(y)| = |y| \leq K_3$. Letting $I = \{(y, j) : j = 1, ...n(y), y \in \mathcal{G}_2\}$ yields the conclusion.

Proof of lemma 3.2(b) From Part (a) we know that B can be partioned into $B = \bigcup_{i \in I} B^i$ such that μ is integer consistent for B^i and $|B^i| \leq K_3$. For each $a \in B_1$ we now choose $i_a \in I$ such that $a \in B^{i_a}$. Define $B_2 = \bigcup_{a \in B_1} B^{i_a}$. Since μ is integer consistent for B^{i_a}, each $a \in B_1$, then μ is integer consistent for B_2. Moreover, since $|B^{i_a}| \leq K_3$ we obtain $|B_2| \leq K_3\, |B_1|$. ■

The following Lemma appears in EGSZ (1999), but we include it here for convenience. This lemma allows us to construct upper and lower bounds for list prices, is used in the proof of Theorem 5.3. By analogy with a notion from cooperative game theory, we say that a subset $L \subset \mathbf{Lists}_M$ is *strictly balanced* if there are strictly positive real numbers $\{\epsilon_L(\ell) : \ell \in L\}$ (called *balancing weights*) such that $\sum_{\ell \in L} \epsilon_L(\ell)\ell \in \mathbf{Cons}$.

Lemma 6.2 *There is a constant $R^* > 0$ such that: If $L \subset \mathbf{Lists}_M$ is a strictly balanced collection and $q \in \mathbf{Trans}$ is a pure transfer then*

$$\max_{\ell \in L} q \cdot \ell \ \geq \ -R^* \min_{\ell \in L} q \cdot \ell$$

Proof: For each strictly balanced collection of lists L, fix balancing weights $\epsilon_L(\cdot)$. Set

$$R^* = \inf\{\epsilon_L(\ell) : L \text{ is a strictly balanced collection }, \ell \in L\}$$

By definition, balancing weights are strictly positive. Because the set of strictly balanced collection of lists is finite, it follows that $R^* > 0$.

To see that R^* has the desired property, fix a strictly balanced collection L with associated balancing weights $\epsilon_L(\cdot)$. Observe that

$$\sum_{\ell \in L} \epsilon_L(\ell) q \cdot \ell = q \cdot \sum_{\ell \in L} \epsilon_L(\ell)\ell = 0$$

Set $L_+ = \{\ell \in L : q \cdot \ell \geq 0\}$ and $L_- = L \setminus L_+$. Collect L_+ terms on the left hand side and L_- terms on the righthand side to obtain:

$$\sum_{\ell \in L_+} \epsilon_L(\ell) q \cdot \ell = -\sum_{q \in L_-} \epsilon_L(\ell) q \cdot \ell \tag{8}$$

Because the coefficients $\epsilon_L(\ell)$ are positive and sum to 1, we have:

$$\max_{\ell \in L} q \cdot \ell = \left[\sum_{\ell \in L} \epsilon_L(\ell)\right] \max_{\ell \in L} q \cdot \ell \geq \left[\sum_{\ell \in L_+} \epsilon_L(\ell)\right] \max_{\ell \in L_+} q \cdot \ell \geq \sum_{\ell \in L_+} \epsilon_L(\ell) q \cdot \ell \tag{9}$$

Because $q \cdot \ell < 0$ for each $\ell \in L_-$, we have:

$$-\sum_{q \in L_-} \epsilon_L(\ell) q \cdot \ell \geq -\min_{\ell \in L_-} \epsilon_L(\ell) \min_{\ell \in L_-} q \cdot \ell \geq -\min_{\ell \in L} \epsilon_L(\ell) \min_{\ell \in L} q \cdot \ell \tag{10}$$

Combining (8), (9) and (10) and recalling the definition of R^* yields the desired inequality. ■

Proof of Theorem 4.3: Let $f^* \equiv (x^*, \mu^*) \in \mathbf{C}_\varepsilon(\mathcal{E})$.

Step 1 For each agent $a \in A$ consider the preferred sets

$$\varphi(a) = \{(x, \ell) \in X_a \ : \ u_a(x, \ell) > u_a(x_a^*, \mu_a^*) \ \}$$

and the corresponding net prefered set

$$\psi(a) = \{(z, \ell) \in \mathbb{R}^N \times \mathbb{R}^{\mathcal{M}} : (z + e_a - \tau(\ell), \ell) \in \varphi(a)\}.$$

Moreover let $\Psi(a) = \psi(a) \cup \{0\}$ for all $a \in A$ and let

$$Z = \sum_{a \in A} \Psi(a) \tag{11}$$

Step 2 Let $K_4 = K_1(N + |\mathcal{M}|)M + K_2 + N + |\mathcal{M}|$, where K_1, K_2 are the constants defined in Lemma 3.1. Define

$$C^* = \{(x, \mu) \in \mathbb{R}^N \times \mathbb{R}^{\mathcal{M}} : x + (WK_4 + \varepsilon|A|)\, \mathbf{1} << -WK_1 \mathrm{dist}\, (\mu, \mathbf{Cons})\, \mathbf{1}\}$$

Note that C^* is a convex cone with origin $(-(WK_4 + \varepsilon|A|)\mathbf{1}, 0)$ and that $0 \in Z$. We want to separate Z from C^*; to accomplish this, we must show that $C^* \cap \mathrm{conv}\, Z = \emptyset$. We suppose not and construct a coalition that can ε-capitablock.

Assume $(z, \mu) \in C^* \cap \mathrm{conv}\, Z$. Hence, by the Shapley-Folkman theorem, we can choose elements $(z_a, \mu_a) \in \mathrm{conv}\, \Psi(a)$ for each $a \in A$ such that:

(i) $(z, \mu) = \sum_{a \in A} (z_a, \mu_a)$

(ii) $|\{a \in A : (z_a, \mu_a) \notin \Psi(a)\}| \leq N + |\mathcal{M}|$

Write

$$A' = \{a \in A : (z_a, \mu_a) \in \Psi(a)\}, \qquad D = \{a \in A : (z_a, \mu_a) \notin \Psi(a)\}$$

Agents $a \in D$ get bundles in the convex hull of $\Psi(a)$; hence $\mu_a \in \mathrm{conv}\mathbf{Lists}_M$. Thus, since there are at most $N + |\mathcal{M}|$ such agents, it follows that:

$$\mathrm{dist}\,(\mu_{A'}, \mathbf{Cons}) \leq (N + |\mathcal{M}|)M + \mathrm{dist}\,(\mu_A, \mathbf{Cons})$$

We can therefore use Lemma 3.1 to choose a subset $A'' \subset A'$ such that

$$\mu_{A''} \in \mathbf{Cons}^*$$

and

$$|A' \setminus A''| \leq K_1((N + |\mathcal{M}|)M + \mathrm{dist}\,(\mu_A, \mathbf{Cons})) + K_2$$

Thus

$$|A \setminus A''| \leq K_1((N + |\mathcal{M}|)M + \mathrm{dist}\,(\mu_A, \mathbf{Cons})) + K_2 + N + |\mathcal{M}| \tag{12}$$

Now define the coalition

$$B = \{a \in A'' : (z_a, \mu_a) \neq (0, 0)\}.$$

We assert that $B \neq \emptyset$ and that B can ε-capitablock. First notice $\mu_B = \mu_{A''} \in \mathbf{Cons}^*$. Moreover, $u_b(z_b + e_b - \tau(\mu_b), \mu_b) > u_b(x_b^*, \mu_b^*)$ for all $b \in B$, since $(z_b, \mu_b) \in \psi(b)$.

We shall now show that that $z_B << -\varepsilon|A|$, which yields that $B \not= \emptyset$, and that B can ε-capitablock. Since $(z, \mu) = \sum_{a \in A}(z_a, \mu_a) \in C^*$,

$$\sum_{a \in A} z_a << [-(WK_4 + \varepsilon|A|) - WK_1 \mathrm{dist}\,(\mu_A, \mathbf{Cons})]\mathbf{1} \tag{13}$$

For each $a \in A$, $z_a \geq -e_a \geq -W\mathbf{1}$, which together with (12) yield

$$\sum_{a \in A \setminus A''} z_a \geq -(K_1((N+|\mathcal{M}|)M + \mathrm{dist}\,(\mu_A, \mathbf{Cons})) + K_2 + N + |\mathcal{M}|)\, W\mathbf{1}.$$

Hence we obtain by (13), using that $K_4 = K_1(N+|\mathcal{M}|)M + K_2 + N + |\mathcal{M}|$,

$$z_B = \sum_{b \in B} z_b = \sum_{a \in A''} z_a = \sum_{a \in A} z_a - \sum_{a \in A \setminus A''} z_a << -\varepsilon|A|\mathbf{1},$$

as required. We conclude that $C^* \cap \mathrm{conv}\, Z = \emptyset$, as claimed.

Step 3 We now use the separation theorem to find prices $(p, q^*) \in \mathbb{R}^N \times \mathbb{R}^{\mathcal{M}}$, $(p, q^*) \neq (0,0)$ and a real number σ such that

$$\begin{aligned} (p, q^*) \cdot (z, \mu) \leq \sigma \qquad & \text{for each } (z, \mu) \in C^* \\ (p, q^*) \cdot (z, \mu) \geq \sigma \qquad & \text{for each } (z, \mu) \in Z \end{aligned}$$

Because $0 \in Z$ we have $\sigma \leq 0$.Since C^* contains a translate of $-\mathbb{R}^N_{++} \times \{0\}$, it follows that that $p \geq 0$. Because C^* contains a translate of $\{0\} \times \mathbf{Cons}$, it follows that q^* vanishes on **Cons** and hence that $q^* \in \mathbf{Trans}$. We claim that $p \neq 0$. To see this, suppose to the contrary that $p = 0$. By construction, $(p, q^*) \neq (0,0)$ so $q^* \neq 0$. Hence there is a $\bar{\mu} \in \mathbb{R}^{\mathcal{M}}$ such that $q^* \cdot \bar{\mu} > 0$. For $\delta > 0$ sufficiently small, $((-(WK_4 + \varepsilon|A|)\mathbf{1}, 0) + (-\mathbf{1}, \delta\bar{\mu})) \in C^*$, so that $(p, q^*) \cdot ((-(WK_4 + \varepsilon|A|) - 1)\mathbf{1}, \delta\bar{\mu}) \leq \sigma \leq 0$. However

$$(p, q^*) \cdot ((-(WK_4 + \varepsilon|A|) - 1)\mathbf{1}, \delta\bar{\mu}) =$$

$$(0, q^*) \cdot ((-(WK_4 + \varepsilon|A|) - 1)\mathbf{1}, \delta\bar{\mu}) = \delta q^* \cdot \bar{\mu}$$

which, by our choice of $\bar{\mu}$, is positive. This is a contradiction, so we conclude that $p \neq 0$, as asserted.

Normalize p such that $p \in \Delta$. Given (p, q^*) and the state (x^*, μ^*), define membership prices q by

$$q_m = q^*_m + \frac{1}{|\pi|}\, p \cdot \mathbf{inp}(\pi, \gamma)$$

for each $m = (\omega, \pi, \gamma) \in \mathcal{M}$. Remark that for an arbitrary $(y, \ell) \in \mathbb{R}^N \times$ **Lists**$_M$

$$(p, q) \cdot (y, \ell) = (p, q^*) \cdot (y + \tau(\ell), \ell).$$

Step 4: Let $K = WK_4$ and let $\phi = \frac{K}{|A|} + \varepsilon$. We now show that (x^*, μ^*) can be ϕ-decentralized using the prices (p, q). Clearly, each club type has balanced budget at the prices (p, q^*) since $q^* \in$ **Trans**. Moreover, notice, that since $((-(WK_4 + \varepsilon|A|)\mathbf{1}, 0)$ belongs to the closure of C^*, it follows that

$$\sigma \geq (p, q^*) \cdot (-(WK_4 + \varepsilon|A|)\mathbf{1}, 0) = -(WK_4 + \varepsilon|A|). \tag{14}$$

Now let

$$E_1 = \{a \in A \mid (p, q) \cdot (x_a^*, \mu_a^*) > p \cdot e_a\}$$

and $E_2 = A \setminus E_1$. Because (x^*, μ^*) is feasible for A and $q^* \in$ **Trans**,

$$(p, q) \cdot (x_A^*, \mu_A^*) = p \cdot (x_A^* + \tau(\mu_A^*)) = p \cdot e_A$$

Since $A = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, it follows that

$$(p, q) \cdot (x_{E_1}^*, \mu_{E_1}^*) - p \cdot e_{E_1} = \; -((p, q) \cdot (x_{E_2}^*, \mu_{E_2}^*) - p \cdot e_{E_2}) \tag{15}$$

The net trade $(x_a^* + \tau(\mu_a^*) - e_a, \mu_a^*)$ is in the closure of $\Psi(a)$ for each a. Using the separation property of prices and (14) we obtain:

$$[(p, q) \cdot (x_{E_2}^*, \mu_{E_2}^*) - p \cdot e_{E_2}\ ] \geq -(WK_4 + \varepsilon|A|) \tag{16}$$

Using (15) and (16) and keeping in mind that expenditure minus income is positive for agents in E_1 and no others yields:

$$\begin{aligned}
\rho^1(f^*, p, q) &= \frac{1}{|A|} \sum_{a \in A} \rho_a^1(f^*, p, q) \\
&= \frac{1}{|A|} \sum_{a \in E_1} [(p, q) \cdot (x_a^*, \mu_a^*) - p \cdot e_a] \\
&= -\frac{1}{|A|} [(p, q) \cdot (x_{E_2}^*, \mu_{E_2}^*) - p \cdot e_{E_2}\ ] \\
&\leq \frac{WK_4}{|A|} + \frac{\varepsilon|A|}{|A|} \; = \frac{K}{|A|} + \varepsilon \; = \phi
\end{aligned}$$

This establishes the budget estimate for the ϕ-decentralization.

To estimate ρ^2, let E_3 be the set of agents for whom there is $(y_a, \upsilon_a) \in B(a, p, q)$ such that $u_a(y_a, \upsilon_a) > u_a(x_a^*, \mu_a^*)$. As before, separation implies

$$(p, q) \cdot [(y_{E_3}, \upsilon_{E_3}) - (e_{E_3}, 0)] \geq -(WK_4 + \varepsilon|A|)$$

Rearranging yields

$$\frac{1}{|A|}(p, q) \cdot [(e_{E_3}, 0) - (y_{E_3}, \upsilon_{E_3})] \leq \frac{WK_4}{|A|} + \frac{\varepsilon|A|}{|A|} = \frac{K}{|A|} + \varepsilon = \phi$$

This establishes the second estimate for the ϕ-decentralization.■

Proof of Theorem 5.3

Step 1 For each positive integer k greater than $\frac{1}{W}$ construct a perturbed economy $\mathcal{E}^k$ by adjoining to the agent set A a single agent of each external characteristic $\omega \in \Omega$. That is, the set of agents in the perturbed economy is:

$$\bar{A} = A \cup \{a_\omega : \omega \in \Omega\}$$

External characteristics, choice sets, endowments and utility functions for agents in A are as in the original economy $\mathcal{E}$. For the added agent a_ω define external characteristics, choice sets, endowments and utility functions by:

$$\begin{aligned} \omega_{a_\omega} &= \omega \\ X_{a_\omega} &= \mathbb{R}_+^N \times \{\ell \in \mathbf{Lists}_M : \ell(\omega', \pi, \gamma) = 0 \text{ if } \omega' \neq \omega\} \\ e_{a_\omega} &= \tfrac{1}{k}\mathbf{1} \\ u_{a_\omega}(x, \ell) &= |x| \end{aligned}$$

Notice, that the endowments of the added agents depend on k. However to avoid notation we do not explicitly refere to k in the notation e_{a_ω}.

Step 2 The demand functions of the added agents are such that, for commodity prices near the boundary of the simplex and for membership prices that are large in absolute value, aggregate excess demand for commodities will be impossibly large. As a consequence, we can write down compact price sets that contain an equilibrium price for the convexification of the economy $\mathcal{E}^k$. Choose a real number $\varepsilon^k > 0$ so small that

$$\left[1-(N-1)\varepsilon^k\right]\left[\frac{|\Omega|}{kN\varepsilon^k} - W(|A|+|\Omega|)\right] - \varepsilon^k(N-1))W(|A|+|\Omega|) > 0.$$

Define a price simplex for private goods

$$\Delta_{\varepsilon^k} = \{p \in \mathbb{R}^N_+ : \sum_{n\in N} p_n = 1 \text{ and } p_n \geq \varepsilon^k \text{ for each } \ n\}.$$

Having chosen ε^k, choose a real number $R^k > 0$ so big that $|\tau(\ell)| \leq \frac{R^k}{2M^*}$ for all $\ell \in \mathbf{Lists}_M$ and

$$\left[1-\varepsilon^k(N-1)\right]\left[\frac{R^k}{2NM^*} - W(|A|+|\Omega|)\right] - \varepsilon^k(N-1)W(|A|+|\Omega|) > 0.$$

Define a bounded set for club transfer prices:

$$Q_{R^k} = \{q \in \mathbf{Trans} : |q_m| \leq R^k \text{ for all } \ m \in \mathcal{M}\}.$$

Step 3 We now define the excess demand correspondence for the economy $\mathcal{E}^k$. Let $p \in \Delta_{\varepsilon^k}, q \in Q_{R^k}$. For each agent $a \in \bar{A}$, write

$$B(a,p,q) = \{(x,\ell) \in X_a : p\cdot x + q\cdot \ell + p\cdot\tau(\ell) \leq p\cdot e_a\}.$$

This is agent a's budget set, assuming that he must pay an equal share of the inputs to club activities. Since we have assumed that $0 \in X_a$ we obtain $B(a,p,q) \neq \emptyset$. Moreover, $B(a,p,q)$ is compact.

Now let

$$\begin{array}{rcll} d(a,p,q) & = & \arg\max\{u_a(x,\ell) : (x,\ell) \in B(a,p,q)\} & \text{and} \\ \zeta(a,p,q) & = & \{(x+\tau(\ell),\ell) - (e_a,0) : (x,\ell) \in d(a,p,q)\} & \end{array}$$

be agent a's demand set and excess demand set. Excess demand sets are uniformly bounded because endowments are bounded, private good prices are bounded away from 0 and club transfer prices are bounded above and below. Since endowments are assumed to be desirable, it is easily checked that the correspondence $(p, q) \to \zeta(a, p, q)$ is upper hemi-continuous for each a. Define the aggregate excess demand correspondence $Z : \Delta_{\varepsilon^k} \times Q_{R^k} \to \mathbb{R}^N \times \mathbb{R}^{\mathcal{M}}$ as

$$Z(p, q) = \sum_{a \in \bar{A}} \zeta(a, p, q) .$$

The correspondence Z is upper hemi-continuous and has compact and non-empty values.

Step 4 We now find a fixed point of the convexified excess demand correspondence. Individual income comes from selling endowments and (perhaps) receiving subsidies for club memberships. The value of each individual's endowment is bounded by W Because club transfer prices lie in the interval $[-R^k, +R^k]$ and individuals can choose no more than M club memberships, subsidies for club memberships are bounded by MR^k. Because private good prices are bounded below by ε^k, individual demand for each private good is bounded above by $\frac{1}{\varepsilon^k}(W + R^k M)$, and individual excess demand for each private good lie between $-W$ and $\frac{1}{\varepsilon^k}(W + R^k M)$. Hence aggregate excess demand for private goods lies in the compact, convex set

$$Z^k = \{z \in \mathbb{R}^N : -W(|A| + |\Omega|) \leq z_n \leq \frac{1}{\varepsilon^k}(W + R^k M)(|A| + |\Omega|) \text{ for each } n\}$$

and aggregate demands for club memberships lie in the compact, convex set

$$C = \{\bar{\mu} \in \mathbb{R}^{\mathcal{M}} : 0 \leq \sum_{m \in \mathcal{M}} \bar{\mu}(m) \leq M(|A| + |\Omega|)\}$$

Now, define the correspondence

$$\Phi : \Delta_{\varepsilon^k} \times Q_{R^k} \times Z^k \times C \to \Delta_{\varepsilon^k} \times Q_{R^k} \times Z^k \times C$$

by

$$\Phi(p, q, z, \mu) = [\operatorname{argmax}\{(p', q') \cdot (z, \mu) : (p', q') \in \Delta_{\varepsilon^k} \times Q_{R^k}\}] \times \operatorname{conv} Z(p, q)$$

It is easily checked that Φ is an upper hemi-continuous correspondence, and that its values are non-empty convex sets. Hence Kakutani's fixed point theorem guarantees that Φ has a fixed point. Thus there is a price pair $(p^k, q^k) \in \Delta_{\varepsilon^k} \times Q_{R^k}$ and a pair $(z^k, \mu^k) \in \operatorname{conv} Z^k(p^k, q^k)$ such that

$$(p^k, q^k) \cdot (z^k, \mu^k) = \max\{(p', q') \cdot (z^k, \mu^k) \,:\, (p', q') \in \Delta_{\varepsilon^k} \times Q_{R^k}\}.$$

Walras' Law implies $(p^k, q^k) \cdot (z^k, \mu^k)$=0

Step 5 We show in several steps that $z^k = 0$ and $\mu^k \in \mathbf{Cons}$.
Step 5.1 We show first that $q^k \cdot \mu^k = 0$. Suppose that this is not so; we obtain a contradiction by looking at excess demands at prices (p^k, q^k) of the added agents in $\bar{A} \setminus A$. Because $0 \cdot \mu^k = 0$ maximality and the definition of Φ entail that $q^k \cdot \mu^k > 0$. Moreover maximality entails that $q^k \in \operatorname{bdy} Q_{R^k}$ so that $|q^k_m| = R^k$ for some $m \in \mathcal{M}$. If $q^k_m = R^k$, then since $q^k \in \mathbf{Trans}$, some other price must have large magnitude and be negative. Thus there is always a membership $m^* = (\omega^*, \pi^*, \gamma^*)$ such that $q^k_{m^*} \leq -R^k/M^*$. The agent $a_{\omega^*} \in \bar{A} \setminus A$ could obtain a subsidy of R^k/M^* by choosing one membership m^* and no other. Remember R^k has been chosen such that $|\tau(\ell)| \leq \frac{R^k}{2M^*}$ for each $\ell \in \mathbf{Lists}_M$. Hence, this agent at most uses the wealth $\frac{R^k}{2M^*}$ for paying for the inputs. Since he does not care about club memberships and finds all private goods to be perfect substitutes, it follows that his excess demand for one of the least expensive private goods -which we may as well suppose to be good 1- is at least $\frac{R^k}{2NM^*} - \frac{1}{k} \geq \frac{R^k}{2NM^*} - W$. Moreover his excess demand for private commodities is convex. Keeping in mind that the convexified individual excess demand for all agents are bounded below by $-W\mathbf{1}$, that the total number of agents in $\bar{A}$ is $|\Omega|+|A|$, and that $(z^k, \mu^k) \in \operatorname{conv} Z(p^k, q^k)$ we obtain:

$$\begin{aligned} z_1^k &\geq \frac{R^k}{2NM^*} - W(|A| + |\Omega|) \\ z_n^k &\geq -W(|A| + |\Omega|) \qquad\qquad \text{if } \; n > 1 \end{aligned}$$

Define $\hat{p} \in \Delta_{\varepsilon^k}$ by:

$$\begin{aligned} \hat{p}_1 &= 1 - \varepsilon^k(N-1) \\ \hat{p}_n &= \varepsilon^k \qquad\qquad \text{if } n > 1 \end{aligned}$$

Calculation shows that

$$\hat{p} \cdot z^k \geq \left[1 - \varepsilon^k(N-1)\right] \left[\frac{R^k}{2NM^*} - W(|A| + |\Omega|)\right] - \varepsilon^k(N-1)W(|A| + |\Omega|)$$

Our choices of R^k, ε^k guarantee that the left hand side of the inequality is strictly positive so

$$(\hat{p}, 0) \cdot (z^k, \mu^k) > 0 = (p^k, q^k) \cdot (z^k, \mu^k)$$

which contradicts maximality. We conclude that $q^k \cdot \mu^k = 0$, as desired.

Step 5.2 $\mu^k \in$ **Cons**. If not, we could find a pure transfer $q' \in$ **Trans** such that $q' \cdot \mu^k > 0$ and hence could find a $q'' \in Q_{R^k}$ such that $q'' \cdot \mu^k > 0$, contradicting maximality.

Step 5.3 $p_n^k > \varepsilon^k$ for each n. Suppose not; we once again obtain a contradiction by considering the excess demand of the added agents in $\bar{A} \setminus A$. No such agent cares about club memberships and finds all commodities to be perfect substitutes. Since these agents have endowment $\frac{1}{k}\mathbf{1}$ they have at least the wealth $\frac{1}{k}$ for buying private commodities for consumption. Since they only consume the least expensive commodities, then for all $u \in \sum_{a \in \bar{A} \setminus A} \zeta(a, p^k, q^k)$, there is at least one commodity h, for which

$$u_h \geq \frac{|\Omega|}{kN\varepsilon^k} - \frac{|\Omega|}{k} \geq \frac{|\Omega|}{kN\varepsilon^k} - W|\Omega| \tag{17}$$

Summing over all agents and keeping in mind that individual excess demands are bounded below by $-W\mathbf{1}$, we conclude that for some commodity, say, commodity 1,

$$\begin{aligned} z_1^k &\geq \frac{|\Omega|}{kN\varepsilon^k} - W(|A| + |\Omega|) \\ z_n^k &\geq -W(|A| + |\Omega|) \qquad\qquad \text{if } n > 1 \end{aligned}$$

Define $\hat{p} \in \Delta_{\varepsilon^k}$ by:

$$\begin{array}{rcll} \hat{p}_1 & = & 1 - \varepsilon^k(N-1) & \\ \hat{p}_n & = & \varepsilon^k & \text{if } n > 1 \end{array}$$

Calculation gives

$$\hat{p} \cdot z^k \geq \left[1 - \varepsilon^k(N-1)\right] \left[\frac{|\Omega|}{kN\varepsilon^k} - W(|A| + |\Omega|)\right] - \varepsilon^k(N-1)W(|A| + |\Omega|)$$

Our choice of ε^k guarantees that the right hand side of the inequality is positive and hence that

$$(\hat{p}, 0) \cdot (z^k, \mu^k) > 0 = (p^k, q^k) \cdot (z^k, \mu^k)$$

which again contradicts maximality. We conclude that $p_n^k > \varepsilon^k$ for each n.

Step 5.4 We show that $z^k = 0$. Notice that $(p^k, q^k) \cdot (z^k, \bar{\mu}^k) = 0$ and $q^k \cdot \bar{\mu}^k = 0$ so $p^k \cdot z^k = 0$. Hence, if $z^k \neq 0$ there are indices i, j such that $z_i^k < 0$ and $z_j^k > 0$. Since $p_i^k > \varepsilon^k$, we can construct a price $\hat{p} \in \Delta_{\varepsilon^k}$ by setting

$$\begin{array}{rcll} \hat{p}_i & = & p_i^k - \frac{1}{2}(p_i^k - \varepsilon^k) & \\ \hat{p}_j & = & p_j^k + \frac{1}{2}(p_i^k - \varepsilon^k) & \\ \hat{p}_n & = & p_n^k & n \neq i, j \end{array}$$

Since $p^k \cdot z^k = 0$, it follows that $\hat{p} \cdot z^k > 0$, a contradiction to maximality. We conclude that $z^k = 0$.

Step 6 In this step we find a subset B^k of A and a feasible state $(y_a^k, \nu_a^k)_{a \in \bar{A}}$ such that $|A \backslash B^k| \leq S$, where S is defined in (21). All agents $a \in B^k$ are in their demand sets. Moreover, the agents in $\bar{A} \backslash B^k$ are not assigned to clubs, and they satisfy their budget constraint in aggregate.

As $(z^k, \mu^k) = (0, \mu^k) \in \text{conv} Z(p^k, q^k)$, we can apply the Shapley-Folkman theorem to obtain $(z_a^k, \mu_a^k) \in \text{conv}\zeta(a, p^k, q^k)$ for $a \in \bar{A}$ and a set $D^k \subset \bar{A}$ with $|D^k| \leq N + |\mathcal{M}|$ such that $\sum_{a \in \bar{A}} (z_a^k, \mu_a^k) = (0, \mu^k)$ and all agents $a \in \bar{A} \setminus D^k$ are optimizing: Hence for $a \in \bar{A} \backslash D^k$ there exists $(x_a^k, \mu_a^k) \in d(a, p^k, q^k)$ such that $z_a^k = x_a^k + \tau(\mu_a^k) - e_a$. Thus

- $\sum_{a\in D^k}(z_a^k,\mu_a^k)+\sum_{a\in\bar{A}\setminus D^k}(x_a^k+\tau(\mu_a^k)-e_a,\mu_a^k)=(0,\mu^k)$
- $(x_a^k,\mu_a^k)\in \operatorname{argmax}\{u_a(x,\ell):(x,\ell)\in B(a,p^k,q^k)\}$ for $a\in\bar{A}\setminus D^k$;
- $(z_a^k,\mu_a^k)\in \operatorname{conv}\zeta(a,p^k,q^k)$ for $a\in D^k$;
- $|D^k|\le N+|\mathcal{M}|$

Let $\bar{B}^k=A\setminus D^k$ and note that

$$|\bar{B}^k|\ge |A|-(N+|\mathcal{M}|). \tag{18}$$

Moreover

$$|\bar{A}\setminus\bar{B}^k|\le N+|\mathcal{M}|+|\Omega|\le M(N+2|\mathcal{M}|)$$

and by Step 5

$$\mu_{\bar{A}}^k=\mu^k\in\mathbf{Cons}.$$

Hence, as $\mu_a^k\in \operatorname{conv}\mathbf{Lists}_M$ for all $a\in\bar{A}\setminus\bar{B}^k$ we obtain,

$$\operatorname{dist}(\mu_{\bar{B}^k}^k,\mathbf{Cons})\le \operatorname{dist}(\mu_{\bar{B}^k}^k,\mu_{\bar{A}}^k)\le M(N+2|\mathcal{M}|) \tag{19}$$

Now we apply Lemma 3.1 to find a subset $B^k\subset\bar{B}^k$ such that

$$\mu_{B^k}^k \;\in\; \mathbf{Cons}^*$$

and

$$|\bar{B}^k\setminus B^k|\le K_1 \operatorname{dist}(\mu_{\bar{B}^k}^k,\mathbf{Cons})+K_2 \tag{20}$$

where K_1,K_2 are the constants defined in Lemma 3.1. Combining (18), (19) and (20), and letting

$$S=N+|\mathcal{M}|+K_1(M(N+2|\mathcal{M}|))+K_2 \tag{21}$$

we have

$$|B^k|\ge|A|-S$$

We can define the state (y^k, ν^k) for $\mathcal{E}^k$.

$$(y_a^k, \nu_a^k) = \begin{cases} (x_a^k, \mu_a^k) & \text{if} \quad a \in B^k \\ (z_a^k + e_a, 0) & \text{if} \quad a \in \bar{A} \setminus B^k \end{cases}$$

The state (y^k, ν^k) is feasible for $\bar{A}$ since $\nu_{\bar{A}}^k = \mu_{B^k}^k \in \mathbf{Cons}^*$ and $\sum_{a \in \bar{A}}(y_a^k + \tau(\nu_a^k) - e_a) = \sum_{a \in \bar{A} \setminus B^k}(x_a^k + \tau(\mu_a^k) - e_a) + \sum_{a \in B^k}(z_a^k + e_a - e_a) = \sum_{a \in \bar{A}} z_a^k = 0$. However, the state is not necessarily feasible for A, since the aggregate net trade in private goods of the artificial agents might not be zero. Also, the agents in $A \setminus B^k$ are not necessarily in their budget sets as $q^k \cdot \mu_a^k$ might not be zero for such agents. However, since $\mu_{B^k}^k, \mu_{\bar{A}}^k \in \mathbf{Cons}$ and $q^k \in \mathbf{Trans}$, it follows that $q^k \cdot \mu_{\bar{A} \setminus B^k}^k = 0$. Hence, as $(z_a^k, \mu_a^k) \in \operatorname{conv}\zeta(a, p^k, q^k)$, and therefore $p^k \cdot z_a^k + q^k \cdot \mu_a^k$ =0, for all $a \in \bar{A} \setminus B^k$, we obtain that

$$(p^k, q^k) \cdot (\sum_{a \in \bar{A} \setminus B^k} (y_a^k - e_a, 0)) = p^k \cdot \sum_{a \in \bar{A} \setminus B^k} z_a^k = -q^k \cdot \sum_{a \in \bar{A} \setminus B^k} \mu_a^k = 0.$$

Hence, in aggregate the agents in $\bar{A} \setminus B^k$ satisfy the budget constraint.

Step 7 By construction, club transfer prices q^k are bounded by R^k, but R^k depends on k. We now replace the sequence of club transfer prices (q^k) by a bounded sequence $(\bar{q}^k)$ which leads to the same demands.

Passing to a subsequence if necessary, we may assume without loss that for each $\ell \in \mathbf{Lists}_M$ the sequence $(q^k \cdot \ell)$ converges to a limit G_ℓ, which may be finite or infinite. Write:

$$\begin{aligned} L &= \{\ell \in \mathbf{Lists}_M : q^k \cdot \ell \to G_\ell \in \mathbb{R}\} \\ L_+ &= \{\ell \in \mathbf{Lists}_M : q^k \cdot \ell \to +\infty\} \\ L_- &= \{\ell \in \mathbf{Lists}_M : q^k \cdot \ell \to -\infty\} \end{aligned}$$

Choose $\bar{G} \in \mathbf{R}$ so large that $|q^k \cdot \ell| \leq \bar{G}$ for each k, each $\ell \in L$.

Define the linear transformation $T : \mathbf{Trans} \to \mathbb{R}^L$ by $T(q)_\ell = q \cdot \ell$. Write $\operatorname{ran} T = T(\mathbf{Trans}) \subset \mathbb{R}^L$ for the range of T and $\ker T = T^{-1}(0) \subset \mathbf{Trans}$ for the kernel (null space) of T. The fundamental theorem of linear algebra implies that we can choose a subspace $H \subset \mathbf{Trans}$ so that $H \cap \ker T = \{0\}$

and $H + \ker T = \mathbf{Trans}$. Write $T_{|H}$ for the restriction of T to H. Note that $T_{|H} : H \to \operatorname{ran} T$ is a one-to-one and onto linear transformation, so it has an inverse $S : \operatorname{ran} T \to H$. Because S is a linear transformation, it is continuous, so there is a constant K such that $|S(x)| \leq K|x|$ for each $x \in \operatorname{ran} T$.

Let R^* be the constant constructed in Lemma 6.2. Choose k_0 so large that $k \geq k_0$ implies

$$\begin{aligned} q^k \cdot \ell &> +2K\bar{G}M + W \qquad \text{if } \ell \in L_+ \\ q^k \cdot \ell &< -2K\bar{G}M - \frac{W}{R^*} \quad \text{if } \ell \in L_- \end{aligned}$$

Write ST for the composition of S with T. For each $k \geq k_0$ set

$$\bar{q}^k = ST(q^k) - ST(q^{k_0}) + q^{k_0} \in \mathbf{Trans}$$

Because $S, T_{|H}$ are inverses, the composition TS is the identity, so

$$T(\bar{q}^k) = TST(q^k) - TST(q^{k_0}) + T(q^{k_0}) = T(q^k)$$

We assert that for $k > k_0$, $\nu_a^k \notin L_- \cup L_+$ for any $a \in \bar{A}$. This holds for $a \in \bar{A} \setminus B^k$ because $\nu_a^k = 0$. If $a \in B^k$ then $q^k \cdot \nu_a^k \leq W$ (because the value of endowment is bounded by W) so $\nu_a^k \notin L_+$, by construction of L_+. Since $\{\nu_a^k\}$ are strictly balanced and $q^k \in \mathbf{Trans}$, it follows from Lemma 6.2 that $\min_{a \in B^k}\{q^k \cdot \nu_a^k\} \geq -\frac{1}{R^*} \max_{a \in B^k}\{q^k \cdot \nu_a^k\} \geq -\frac{W}{R^*}$, and hence $\nu_a^k \notin L_-$ by the construction of L_-.

We now chose $k_1 \geq k_0$ such that for all $\ell \in L_-$ and all $k > k_1$ we have $q^k \cdot \ell < q^{k_0} \cdot \ell - 2KGM$. We claim that when $k > k_1$ then (y_a^k, ν_a^k) is a maximal element in $B(a, p^k, \bar{q}^k)$ for each agent $a \in B^k$. To check this, keep in mind that individual demands for private goods and club memberships can be thought of as depending only on the prices of private goods and of lists, not directly on the prices of memberships. We showed above that $\nu_a^k \in L$ for $a \in \bar{A}$; by construction $\bar{q}^k \cdot \ell = q^k \cdot \ell$ for all $\ell \in L$ because $T(\bar{q}^k) = T(q^k)$. Hence choices are budget feasible. Suppose now that (y, ℓ) is budget feasible for $a \in B^k$ at prices $(p^k, \bar{q}^k)$ and preferred to (y_a^k, ν_a^k). Budget feasibility of (y, ℓ) at prices $(p^k, \bar{q}^k)$ implies that $\bar{q}^k \cdot \ell \leq W$ and hence $q^{k_0} \cdot \ell \leq W + 2K\bar{G}M$ because $|ST(q^k)| \leq K\bar{G}$ and $|ST(q^{k_0})| \leq K\bar{G}$. Thus $\ell \notin L_+$. For $\ell \in L_-$ and $k > k_1$, we similarly obtain $\bar{q}^k \cdot \ell > q^{k_0} \cdot \ell - 2K\bar{G}M > q^k \ell$. Thus, $\bar{q}^k \cdot \ell \geq q^k \cdot \ell$ for $\ell \in L_-$. Hence $\bar{q}^k \cdot \ell \geq q^k \cdot \ell$ for $\ell \in L_- \cup L$. Thus budget feasibility of (y, ℓ) at prices $(p^k, \bar{q}^k)$ implies budget feasibility of (y, ℓ) at prices (p^k, q^k), so (y_a^k, ν_a^k) is not optimal at prices (p^k, q^k). It follows that (y^k, ν^k) are optimal choices at prices $(p^k, \bar{q}^k)$ for all agents in B^k. By construction, $|\bar{q}^k \cdot \ell| \leq 2K\bar{G}M + |q^{k_0} \cdot \ell|$ for $k > k_0$ and for all lists ℓ. Because singleton memberships are themselves lists, it follows that $(\bar{q}^k)$ is a bounded sequence in **Trans**.

Step 8 We thus have a bounded sequence $(p^k, \bar{q}^k)$. Clearly, also the sequence (y^k) is bounded since the sequence is non-negative and $0 \leq \sum_{a \in \bar{A}} y_a^k \leq W(|A| + |\Omega|)$. Moreover $\nu_a^k \in \mathbf{Lists}_M$ for all a. Passing to subsequences as necessary, we may assume that B^k is constant, say B^*, and that ν^k is constant, say ν^*, along the subsequence, and that $p^k \to p^* \in \Delta$, $\bar{q}^k \to q^* \in \mathbf{Trans}$, $(y^k) \to y^*$.

In order to define a feasible state for $\mathcal{E}$ first notice that we can without loss of generality assume that $A \setminus B^* \neq \emptyset$ (compare Step 6). We now define the state $(\bar{y}^*, \bar{\nu}^*)$ by

$$(\bar{y}_a^*, \bar{\nu}_a^*) = \begin{cases} (y_a^*, \nu_a^*) & \text{if} \quad a \in B^* \\ \left(\left[\frac{p^* \cdot e_a}{p^* \cdot \sum_{a \in A \setminus B^*} e_a} \sum_{a \in \bar{A} \setminus B^*} y_a^* \right], 0 \right) & \text{if} \quad a \in A \setminus B^* \end{cases}$$

We shall show that $(\bar{y}^*, \bar{\nu}^*)$ and the prices (p^*, q) are an S-quasi-equilibrium for $\mathcal{E}$, where S is defined in (21), and where q are the membership prices corresponding to the club transfer prices q^*, i.e., $q(\omega, \pi, \gamma) = q^*(\omega, \pi, \gamma) + p \cdot \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma)$.

Our construction and the fact that $\lim_{k \to \infty} \sum\limits_{a \in \bar{A} \setminus A} e_a = \lim_{k \to \infty} \frac{1}{k} |\Omega| = 0$ yield that $(\bar{y}^*, \bar{\nu}^*)$ is feasible for A. Clearly, $(\bar{y}_a^*, \bar{\nu}_a^*) \in B(a, p^*, q^*)$ for $a \in B^*$. Moreover, as noticed in Step 6, we have $p^k \cdot \sum\limits_{a \in \bar{A} \setminus B^*} y_a^k = p^k \cdot \sum\limits_{a \in \bar{A} \setminus B^k} e_a$ for every

k and hence

$$(p^*, q^*) \cdot (\left[\frac{p^* \cdot e_a}{p^* \cdot \sum\limits_{a \in A \setminus B^*} e_a} \quad \sum\limits_{a \in \bar{A} \setminus B^k} y_a^* \right], 0) = p^* \cdot e_a$$

for $a \in A \setminus B^*$. Hence all agents are in their budget sets at prices (p^*, q^*). We complete the proof by showing that for all $a \in B^*$, if $u_a(y, \ell) > u_a(\bar{y}_a^*, \bar{\nu}_a^*)$ then $(p^*, q^*) \cdot (y + \tau(\ell), \ell) \geq p^* \cdot e_a$. However, this follows since, if $(p^*, q^*) \cdot (y + \tau(\ell), \ell) < p^* \cdot e_a$, then $(p^k, \bar{q}^k) \cdot (y + \tau(\ell), \ell) < p^k \cdot e_a$ for k sufficiently large, which contradicts that $(x_a^k, \mu_a^k) \in d(a, p^k, \bar{q}^k)$. Let $D = A \setminus B^*$. Then since $|D| = |A \setminus B^*| \leq S$, we have shown that the state $(\bar{y}^*, \bar{\nu}^*)$ and the prices (p^*, q) are an S-quasi-equilibrium. ■

Proof of Theorem 5.4: Let S be a non-negative integer, $\epsilon > 0$, and let $\hat{K} = WK_3$ where K_3 is the constant in Lemma 3.2. Let $((x, \mu), (p, q))$ be an S-quasi-equilibrium for an economy $\mathcal{E}$ with $|A| > \frac{\hat{K}S}{\varepsilon} = \frac{WK_3S}{\varepsilon}$. Assume that $B \subset A$ can ε-capitablock. Then there exists (y, ν) such that $u_a(y_a, \nu_a) > u_a(x_a, \mu_a)$ for every $a \in B$, ν is integer consistent for B and $y_B + \tau(\nu_B) \leq e_B - |A|\varepsilon\mathbf{1}$. Since $((x, \mu), (p, q))$ is an S-quasi-equilibrium, there exists $D \subset A$ with $|D| \leq S$ such that if $a \in B \setminus D$ then $(p, q) \cdot (y_a, \nu_a) \geq p \cdot e_a$. Apply Lemma 3.2(b) to ν, to B and to $B_1 \equiv B \cap D \subset B$ to obtain B_2 such that $B_1 \subset B_2$, $|B_2| \leq K_3|B_1| \leq K_3S$, and ν is integer consistent for B_2. Clearly ν is also integer consistent for $C = B \setminus B_2$. Hence, by budget balance for club types we have $q \cdot \nu_C = p \cdot \tau(\nu_C)$ and $q \cdot \nu_{B_2} = p \cdot \tau(\nu_{B_2})$. Thus, as $(p, q) \cdot (y_a, \nu_a) \geq p \cdot e_a$ for agents in C and agents in B_2 are in their budget sets, we obtain $p \cdot y_C + p \cdot \tau(\nu_C) = (p, q) \cdot (y_C, \nu_C) \geq p \cdot e_C$ and $p \cdot y_{B_2} + p \cdot \tau(\nu_{B_2}) = (p, q) \cdot (y_{B_2}, \nu_{B_2}) \geq 0$. Thus for $B = B_2 \cup C$ we obtain

$$p \cdot y_B + p \cdot \tau(\nu_B) \geq p \cdot y_C.$$

As endowments of the agents $a \in B_2$ are bounded by $W\mathbf{1}$ and $|B_2| \leq K_3S$ this inequality yields

$$p \cdot y_B + p \cdot \tau(\nu_B) \geq p \cdot e_B - WK_3S.$$

However, as B ε-capitablocked and $|A| > \frac{WK_3S}{\varepsilon}$ we know

$$p \cdot y_B + p \cdot \tau(\nu_B) \leq p \cdot e_B - \varepsilon|A| < p \cdot e_B - WK_3S..$$

Hence we have a contradiction. Thus $(x, \mu) \in \mathbf{C}_\varepsilon(\mathcal{E})$. ■

7 References

R. M. Anderson, "An Elementary Core Equivalence Theorem", *Econometrica* 46 (1978), 1483-1487.

R. M. Anderson, "Strong Core Theorems with Nonconvex Preferences", *Econometrica* 53 (1985), 1283-1293.

T. Bewley, "A Critique of Tiebout's Theory of Local Public Expenditures," *Econometrica* 49 (1981), 713-740.

J. Buchanan, "An Economic Theory of Clubs," *Economica* 33 (1965), 1-14.

B. Ellickson, "A Generalization of the Pure Theory of Public Goods," *American Economic Review* 63 (1973), 417-432.

B. Ellickson, B. Grodal, S. Scotchmer and W. Zame, "Clubs and the Market", forthcoming *Econometrica* 1999.

B. Ellickson, B. Grodal, S. Scotchmer and W. Zame, "Clubs and the Market: Large Finite Economies," IBER Working Paper 97-255, U.C., Berkeley (April 1997).

B. Grodal, "Existence of Appoximate Cores with Incomplete Preferences", *Econometrica* 44 (1976), 829-830.

B. Grodal and W. Hildenbrand, "Limit Theorems for Approximate Cores", Working Paper IP-208 Center for Research and Management, U.C. Berkeley (1974).

B. Grodal, W. Trockel and S. Weber, "On Approximate Cores of Non-convex Economies", *Economic Letters* 15 (1984), 197-203.

W. Hildenbrand, *Core and Equilibria of a Large Economy*, Princeton University Press, New Jersey (1974).

W. Hildenbrand, D. Schmeidler and S. Zamir, "Existence of Approximate Equilibria and Cores", *Econometrica* 41 (1973), 1159-1166.

Y. Kannai (1970) "Continuity Properties of the Core of a Market", *Econometrica* 38, 791-850, and "A Correction", *Econometrica* 40 (1972), 955-58.

S. Scotchmer, "On Price-Taking Equilibrium in Club Economies with Nonanonymous Crowding," *Journal of Public Economics* 65 (1997), 75-88.

R. Starr, "Quasi Equilibria in Markets with Non-convex Preferences", *Econometrica* 37 (1969), 25-38.